THE MIXED PROBLEM FOR THE ELASTIC ANISOTROPIC HALFPLANE

(SMESHANNAIA ZADACHA DLIA UPRUGOI ANIZOTROPNOI POLUPLOSKOSTI)

PMM Vol.26, No.5, 1962, pp. 896-906

V.Z. SVEKLO
(Kaliningrad)

(Received December 8, 1961)

1. Formulation of the problem. We seek a regular solution, i.e. a solution that is continuous up to second order derivatives of the equations of motion [1]

$$a\frac{\partial^2 u}{\partial x^2}+d\frac{\partial^2 u}{\partial y^2}+c\frac{\partial^2 v}{\partial x\,\partial y}+X=\rho\frac{\partial^2 u}{\partial t^2}$$
$$c\frac{\partial^2 u}{\partial x\,\partial y}+d\frac{\partial^2 v}{\partial x^2}+a\frac{\partial^2 v}{\partial y^2}+Y=\rho\frac{\partial^2 v}{\partial t^2} \qquad (1.1)$$

at points in an anisotropic halfplane $y \geqslant 0$ under the following initial conditions

$$u\,(x,\,y,\,0)=u_0\,(x,\,y),\qquad v\,(x,\,y,\,0)=v_0\,(x,\,y),\qquad (\partial u/\partial t)_0=u_0'\,(x,\,y),$$
$$(\partial v\,/\,\partial t)_0=v_0'\,(x,\,y) \qquad (1.2)$$

and the following boundary conditions:

$$\tau_{xy}\,(x,\,0,\,t)=A\,(x,\,t),\qquad \sigma_y\,(x,\,0,\,t)=B\,(x,\,t) \qquad (1.3)$$

The right sides of these equations contain given functions.

2. The Green-Volterra formula. In the general case of anisotropy we have the Green-Volterra formula

$$\iiint\limits_T (u_1X_2+v_1Y_2-u_2X_1-v_2Y)\,dxdydt=$$
$$=\iint\limits_S [u_1P\,(u_2,\,v_2)+v_1Q\,(u_2,\,v_2)-u_2P\,(u_1,\,v_1)-v_2Q\,(u_1,\,v_1)]\,dS \qquad (2.1)$$

Here u_1, v_1 is a solution of Equations (1.1) corresponding to the body forces X_1, Y_1, while the solution u_2, v_2 corresponds to the body forces X_2, Y_2; T denotes an arbitrary volume in the xyt space which is bounded by the surface S with interior normal n

$$P(u, v) = \sigma_x \cos(nx) + \tau_{xy} \cos(ny) - \rho \frac{\partial u}{\partial t} \cos(nt)$$
$$Q(u, v) = \tau_{xy} \cos(nx) + \sigma_y \cos(ny) - \rho \frac{\partial v}{\partial t} \cos(nt) \tag{2.2}$$

This formula remains valid even when one of the solutions has a true strong discontinuity [2].

3. Fundamental solutions. We shall construct some special solutions of the homogeneous Equations (1.1). These will be nonzero within a characteristic cone whose vertex is at the point (x_0, y_0, t_0) and will have the required singularity on its axis $x = x_0$, $y = y_0$. We call them fundamental solutions. Let us introduce the functions

$$\omega_{1j}^\circ = \gamma\left(\frac{c-d}{c}\frac{\lambda'_j}{\theta_j\lambda_j} - \frac{a}{L_{1j}}\right), \qquad \lambda_j' = \frac{d\lambda_j}{d\theta_j},$$
$$\omega_{2j}^\circ = \gamma\left(\frac{c-d}{L_{1j}} - \frac{a}{c}\frac{\lambda'_j}{\theta_j\lambda_j}\right), \qquad \gamma = \frac{c}{a^2-(c-d)^2}, \qquad L_{1j} = a\theta_j^2 + d\lambda_j^2 - 1 \tag{3.1}$$

Between θ_j and λ_j there exists the relation

$$L_{1j}L_{2j} - c^2\theta_j^2\lambda_j^2 = 0, \qquad L_{2j} = d\theta_j^2 + a\lambda_j^2 - 1$$
$$\delta_j = t_0 - t - (x - x_0)\theta_j + (y - y_0)\lambda_j = 0 \tag{3.2}$$

We specify incident disturbances in the form

$$u_k^{\circ\circ} = \sum_{j=1}^{2} \operatorname{Re} i \int^{\theta_j} c\xi\lambda_j(\xi)\,\omega_{kj}^{\circ\circ}(\xi)\,d\xi, \qquad v_k^{\circ\circ} = \sum_{j=1}^{2} \operatorname{Re} i \int^{\theta_j} L_{1j}(\xi)\,\omega_{kj}^{\circ\circ}(\xi)\,d\xi \tag{3.3}$$

where

$$\omega_{kj}^{\circ\circ} = \frac{L_{2j} - c\lambda_j^2}{F_j^\circ}\,\omega_{kj}^\circ \quad (k = 1, 2), \quad \omega_{3j}^{\circ\circ} = -\frac{\lambda_j}{\theta_j}\frac{c\theta_j^2\omega_{1j}^\circ + L_{2j}\omega_{2j}^\circ}{F_j^\circ}$$
$$F_j^\circ = c^2\lambda_j^2\left(\theta_j^2\omega_{1j}^\circ + \lambda_j^2\omega_{2j}^\circ\right) + L_{2j} - c\lambda_j^2 \tag{3.4}$$

Forming the solutions [3] for the reflected disturbances under the condition that the boundary is stress free, and superposing these solutions on the corresponding incident solutions, we obtain the fundamental solutions u_k°, v_k°. These are nonzero in the interior of a characteristic cone and zero on its boundary and in its exterior. Upon passage through the pertinent characteristic surface, lying either within the cone mentioned above or comprising its surface, the first derivatives of these solutions suffer a discontinuity. However, as is easily shown, the kinematic and dynamic conditions of compatibility are satisfied.

4. Estimate of the fundamental solutions near the axis of the characteristic cone. To make this estimate, we show that it is sufficient to estimate the function (3.3) for large values of θ_j, since only incident disturbances have singularities on the axis of the cone. Taking into account the branch of λ_j to be chosen for $c < a - d$ [1], and writing out only the main terms, we obtain

$$\lambda_j = -M_j i\theta_j, \qquad M_j = \left[\frac{L_0}{2ad} + (-1)^{j+1}\sqrt{\frac{L_0^2}{4a^2d^2} - 1}\right]^{1/2}, \quad L_0 = a^2 + d^2 - c^2 \tag{4.1}$$

whereby

$$M_1 M_2 = 1, \quad M_1^2 + M_2^2 = \frac{L_0}{ad}, \quad (a - M_j^2 d)(d - M_j^2 a) + c^2 M_j^2 = 0 \tag{4.2}$$

From (3.2) we find

$$\theta_j = \frac{x' - iM_j y'}{r_j'^2} t', \quad x' = x - x_0, \quad y' = y - y_0, \quad t' = t_0 - t$$
$$r_j'^2 = x'^2 + M_j^2 y'^2 \tag{4.3}$$

For large θ_j we readily obtain

$$\omega_{1j}^\circ = -\sum_{j=1}^{2} \frac{A_j^\circ}{\theta_j^2} + o(\theta_j^{-2}), \qquad \omega_{2j}^\circ = -\sum_{j=1}^{2} \frac{A_j^\circ}{M_j^2\theta_j^2} + o(\theta_j^{-2})$$
$$A_j^\circ = \frac{\gamma d}{c} \frac{\Pi_j}{a - M_j^2 d}, \qquad \Pi_j = a + (c - d) M_j^2 \qquad (j = 1, 2) \tag{4.4}$$

Hence

$$\lim (\theta_j^2 \omega_{1j}^\circ + \lambda_j^2 \omega_{2j}^\circ) = 0, \qquad \theta_j \to \infty$$

and for large θ_j the main parts of the functions ω_{kj}° and $\omega_{kj}^{\circ\circ}$ coincide. Now it is easy to estimate the fundamental solutions. We obtain

$$u_1^\circ = -c\sum_{j=1}^{2} M_j A_j^\circ \operatorname{Re}\theta_j, \qquad v_1^\circ = -\sum_{j=1}^{2} B_j^\circ \operatorname{Re} i\theta_j, \qquad B_j^\circ = \frac{\gamma d}{c}\Pi_j \tag{4.5}$$

or

$$u_1^\circ = -c\sum_{j=1}^{2} M_j A_j^\circ \frac{x't'}{r_j'^2}, \qquad v_1^\circ = -\sum_{j=1}^{2} M_j B_j^\circ \frac{y't'}{r_j'^2} \tag{4.6}$$

It is also necessary to estimate the derivatives for large θ_j. Using Equations (3.2) and (4.3) we find

$$\frac{\partial u_1{}^\circ}{\partial x} = c \sum_{j=1}^{2} M_j A_j{}^\circ \frac{x'^2 - M_j{}^2 y'^2}{r'^4_j} t' \tag{4.7}$$

The same result may be obtained by straightforward differentiation of the expression for $u_1{}^\circ$ with respect to x. The remaining derivatives can be estimated in an analogous manner. Omitting the details of the calculations, we give the final results of the estimates of the solutions and the corresponding stresses.

a) First fundamental solution. Formulas (4.6) are supplemented by

$$\sigma_{x1}{}^\circ = d \sum_{j=1}^{2} M_j \Pi_j A_j{}^\circ \frac{x'^2 - M_j{}^2 y'^2}{r'^4_j} t', \qquad \sigma_{y1}{}^\circ = -d \sum_{j=1}^{2} \frac{\Pi_j A_j{}^\circ}{M_j} \frac{x'^2 - M_j{}^2 y'^2}{r'^4_j} t' \tag{4.8}$$

$$\tau_{xy_1}{}^\circ = d \sum_{j=1}^{2} M_j \Pi_j A_j{}^\circ \frac{2x'y't'}{r'^4_j}$$

b) Second fundamental solution

$$u_2{}^\circ = -c \sum_{j=1}^{2} M_j{}^{-1} A_j \frac{x't'}{r'^2_j}, \qquad v_2{}^\circ = -\sum_{j=1}^{2} M_j{}^{-1} B_j{}^\circ \frac{y't'}{r'^2_j} \tag{4.9}$$

The stresses are obtained in the form

$$\sigma_{x_2}{}^\circ = d \sum_{j=1}^{2} M_j{}^{-1} \Pi_j A_j{}^\circ \frac{x'^2 - M_j{}^2 y'^2}{r'^4_j} t', \qquad \sigma_{y_2}{}^\circ = -d \sum_{j=1}^{2} M_j{}^{-3} \Pi_j A_j{}^\circ \frac{x'^2 - M_j y'^2}{r'^4_j} t'$$

$$\tau_{xy_2} = d \sum_{j=1}^{2} M_j{}^{-1} \Pi_j A_j{}^\circ \frac{2x'y't'}{r'^4_j} \tag{4.10}$$

c) Third fundamental solution

$$u_3{}^\circ = -c \sum_{j=1}^{2} M_j A_j{}^\circ \frac{y't'}{r'^2_j}, \qquad v_3{}^\circ = \sum_{j=1}^{2} M_j{}^{-1} B_j{}^\circ \frac{x't'}{r'^2_j} \tag{4.11}$$

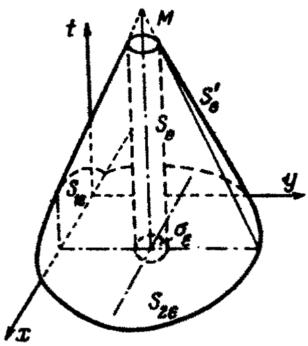
For the stresses we obtain

$$\sigma_{x_3}{}^\circ = d \sum_{j=1}^{2} M_j \Pi_j A_j{}^\circ \frac{2x'y't'}{r'^4_j}, \qquad \sigma_{y_3}{}^\circ = -d \sum_{j=1}^{2} M_j{}^{-1} \Pi_j A_j{}^\circ \frac{2x'y't'}{r'^4_j}$$

$$\tau_{xy_3}{}^\circ = -d \sum_{j=1}^{2} M_j{}^{-1} \Pi_j A_j{}^\circ \frac{x'^2 - M_j{}^2 y'^2}{r'^4_j} t' \tag{4.12}$$

The main parts are indicated in all of the formulas.

5. Solution of the Cauchy problem for the halfplane. To solve this problem, we apply the Green-Volterra formula to the sought solution and to one of the fundamental solutions u_k°, v_k°. For the region of integration we take a portion of the space which is bounded on the one side by the surface S' of a characteristic cone whose vertex is at the point (x_0, y_0, t_0) and, on the other side (see the figure), by a portion S_1 of the plane $y = 0$, and a portion S_2 of the plane $t = 0$, and finally, by a portion of the cylindrical surface S_ε of radius ε that cuts out the axis of the cone. It is to be understood that S' is that portion of the surface of the cone that corresponds to a variation of θ_1 in the interval $-1/\sqrt{a} < \theta_1 < 1/\sqrt{a}$. It is just on this portion that all of the fundamental solutions vanish. On the other hand, taking into consideration that the latter correspond to zero body-force solutions, we obtain



$$\iint_{S'_\varepsilon} B_k dS + \iint_{S_{1\varepsilon}} B_k\, dx\, dt + \iint_{S_{2\varepsilon}} B_k\, dx\, dy + \iint_{S_\varepsilon} B_k\, dS = - \iiint_{T_\varepsilon} (u_k^\circ X + v_k^\circ Y)\, dx\, dy\, dt \tag{5.1}$$

where the index ε indicates that at present (up to passage to the limit), the portion of the aforementioned surfaces that depend on ε are used. The expressions for B_k have the form

$$\begin{aligned} B_k = &\left(\sigma_{x_k}{}^\circ \cos(nx) + \tau_{xy_k}{}^\circ \cos(ny) - \rho \frac{\partial u_k{}^\circ}{\partial t} \cos(nt)\right) u + \\ &+ \left(\tau_{yx_k}{}^\circ \cos(nx) + \sigma_{y_k}{}^\circ \cos(ny) - \rho \frac{\partial v_k{}^\circ}{\partial t} \cos(nt)\right) v - \\ &- \left(\sigma_x \cos(nx) + \tau_{xy} \cos(ny) - \rho \frac{\partial u}{\partial t} \cos(nt)\right) u_k{}^\circ - \\ &- \left(\tau_{xy} \cos(nx) + \sigma_y \cos(ny) - \rho \frac{\partial v}{\partial t} \cos(nt)\right) v_k{}^\circ \end{aligned} \tag{5.2}$$

All of the integrals in Formula (5.1) are known, with the exception of the integral over S_ε. The integral over S'_ε is zero as a consequence of the fact that the kinematic and dynamic compatibility conditions are satisfied for the solutions u_k° and v_k°. The integrals over $S_{2\varepsilon}$ and $S_{1\varepsilon}$ are known by virtue of the initial and boundary conditions. We next show that the integral over S_ε, in the limit as $\varepsilon \to 0$, is equal to a certain linear combination of the derivatives of the unknown functions u and v. On the surface S_ε we have

$$\cos(nt) = 0, \qquad \cos(nx) = \frac{x'}{r'}, \qquad \cos(ny) = \frac{y'}{r'}, \quad r'^2 = x'^2 + y'^2 = \varepsilon^2$$

Setting $x' = \varepsilon \cos \varphi$, $y' = \varepsilon \sin \varphi$, we obtain

$$\iint_{S_\varepsilon} B_k \, dS = \int_0^{t_0 - \eta(\varepsilon)} \Big\{ \int_{L_\varepsilon} [(\sigma_{x_k}{}^\circ \cos\varphi + \tau_{xy_k}{}^\circ \sin\varphi) u + (\tau_{yx_k}{}^\circ \cos\varphi + \sigma_{y_k}{}^\circ \sin\varphi) v +$$
$$+ (\sigma_x \cos\varphi + \tau_{xy} \sin\varphi) u_k{}^\circ - (\tau_{yx} \cos\varphi + \sigma_y \sin\varphi) v_k{}^\circ] \, dl \Big\} dt \quad (5.3)$$

where $\eta(\varepsilon)$ and ε vanish together, and L_ε is a circle of radius ε.

Since the unknown solution is regular, we have, for example,

$$u(x, y, t) = u(x_0 + \varepsilon\cos\varphi, y_0 + \varepsilon\sin\varphi, t) =$$
$$= u(x_0, y_0, t) + \varepsilon\left[\frac{\partial u}{\partial x_0}\cos\varphi + \frac{\partial u}{\partial y_0}\sin\varphi\right] + \ldots$$
$$\sigma_x(x, y, t) = \sigma_x(x_0 + \varepsilon\cos\varphi, y_0 + \varepsilon\sin\varphi, t) =$$
$$= \sigma_x(x_0, y_0, t) + \varepsilon\left[\frac{\partial \sigma_x}{\partial x_0}\cos\varphi + \frac{\partial \sigma_x}{\partial y_0}\sin\varphi\right] + \ldots$$

etc. The omitted terms are of order ε^2 and higher. Substituting into Equation (5.3) and taking into account that $dl = \varepsilon d\varphi$, we obtain

$$\iint_{S_k} B_k \, dS = \int_0^{t_0 - \eta} \varepsilon \Big\{ u(x_0, y_0, t) \int_0^{2\pi} (\sigma_{x_k}{}^\circ \cos\varphi + \tau_{xy_k}{}^\circ \sin\varphi) \, d\varphi +$$
$$+ v(x_0, y_0, t) \int_0^{2\pi} (\tau_{yx_k}{}^\circ \cos\varphi + \sigma_{y_k}{}^\circ \sin\varphi) \, d\varphi +$$
$$+ \frac{\partial u}{\partial x_0} \varepsilon \int_0^{2\pi} (\sigma_{x_k}{}^\circ \cos^2\varphi + \tau_{xy}{}^\circ \sin\varphi\cos\varphi) \, d\varphi +$$
$$+ \frac{\partial u}{\partial y_0} \varepsilon \int_0^{2\pi} (\sigma_{x_k}{}^\circ \sin\varphi\cos\varphi + \tau_{xy_k}{}^\circ \sin^2\varphi) \, d\varphi +$$
$$+ \frac{\partial v}{\partial x_0} \varepsilon \int_0^{2\pi} (\tau_{yx_k}{}^\circ \cos^2\varphi + \sigma_{y_k}{}^\circ \sin\varphi\cos\varphi) \, d\varphi +$$
$$+ \frac{\partial v}{\partial y_0} \varepsilon \int_0^{2\pi} (\tau_{yx_k}{}^\circ \sin\varphi\cos\varphi + \sigma_{y_k}{}^\circ \sin^2\varphi) \, d\varphi -$$

$$- \varepsilon \int_0^{2\pi} [(\sigma_x{}^\circ \cos\varphi + \tau_{xy}{}^\circ \sin\varphi)\, u_k{}^\circ + (\tau_{xy}\cos\varphi + \sigma_y{}^\circ \sin\varphi)\, v_k{}^\circ]\, d\varphi\Big\}\, dt$$

$$\sigma_x{}^\circ = \sigma_x(x_0, y_0, t), \ldots \tag{5.4}$$

Terms which vanish when ε vanishes have been omitted. In the sequel, we shall need the integrals

$$\int_0^{2\pi} \frac{\sin\varphi\cos\varphi\, d\varphi}{r_{j\varphi}^2} = \int_0^{2\pi} \frac{\sin\varphi\cos\varphi\, d\varphi}{r_{j\varphi}^4} = \int_0^{2\pi} \frac{\sin\varphi\cos^3\varphi}{r_{j\varphi}^4}\, d\varphi = \int_0^{2\pi} \frac{\cos\varphi\sin^3\varphi}{r_{j\varphi}^4}\, d\varphi = 0$$

$$\int_0^{2\pi} \frac{\cos^4\varphi - M_j^2\sin^2\varphi\cos^2\varphi}{r_{j\varphi}^4}\, d\varphi = \frac{2\pi}{(1+M_j)^2}$$

$$\int_0^{2\pi} \frac{\sin^2\varphi\cos^2\varphi - M_j^2\sin^4\varphi}{r_{j\varphi}^4}\, d\varphi = -\frac{2\pi}{(1+M_j)^2} \tag{5.5}$$

$$\int_0^{2\pi} \frac{\sin^2\varphi\cos^2\varphi}{r_{j\varphi}^4}\, d\varphi = \frac{\pi}{M_j(1+M_j)^2}, \qquad \int_0^{2\pi} \frac{\sin\varphi\cos^2\varphi}{r_{j\varphi}^4}\, d\varphi = \int_0^{2\pi} \frac{\cos\varphi\sin^2\varphi}{r_{j\varphi}^4}\, d\varphi = 0$$

$$r_{j\varphi}^2 = \cos^2\varphi + M_j^2\sin^2\varphi$$

It is easy to convince oneself that in Equation (5.4) the coefficients of $u(x_0, y_0, t)$ and $v(x_0, y_0, t)$ are equal to zero for all of the fundamental solutions. In addition, as a consequence of Equation (5.5), the following integrals are zero for the first and second fundamental solutions

$$\int_0^{2\pi} (\sigma_{x_k}{}^\circ \sin\varphi\cos\varphi + \tau_{xy_k}{}^\circ \sin^2\varphi)\, d\varphi = \int_0^{2\pi} (\tau_{yx_k}{}^\circ \cos^2\varphi + \sigma_{y_k}{}^\circ \sin\varphi\cos\varphi)\, d\varphi = 0$$

$$(k = 1, 2)$$

Conversely, the following integral vanishes for the third fundamental solution.

$$\int_0^{2\pi} (\sigma_{x_3}{}^\circ \cos^2\varphi + \tau_{xy_3}{}^\circ \sin\varphi\cos\varphi)\, d\varphi = \int_0^{2\pi} (\tau_{yx_3}{}^\circ \sin\varphi\cos\varphi + \sigma_{y_3}{}^\circ \sin^2\varphi)\, d\varphi = 0$$

Let us carry out the calculation of the first fundamental solution in more detail. Denoting the known quantities by $D_{1\varepsilon}$, we write

$$\iint_{S_\varepsilon} B_1\, dS = \int_0^{t_0 - \tau_1(\varepsilon)} \Big\{\frac{\partial u}{\partial x_0}\, \varepsilon^2 \int_0^{2\pi} (\sigma_{x_1}{}^\circ \cos^2\varphi + \tau_{xy_1}{}^\circ \sin\varphi\cos\varphi)\, d\varphi +$$

$$+\frac{\partial v}{\partial y_0}\varepsilon^2\int_0^{2\pi}(\tau_{xy_1}{}^\circ\sin\varphi\cos\varphi+\sigma_{y_1}{}^\circ\sin^2\varphi)\,d\varphi-$$

$$-\varepsilon\int_0^{2\pi}[\sigma_x(x_0,y_0,t)\cos\varphi+\tau_{xy}(x_0,y_0,t)\sin\varphi]\,u_1{}^\circ d\varphi- \tag{5.6}$$

$$-\varepsilon\int_0^{2\pi}[\tau_{xy}(x_0,y_0,t)\cos\varphi+\sigma_y(x_0,y_0,t)\sin\varphi]\,v_1{}^\circ d\varphi\Big\}\,dt=D_{1\varepsilon}+\eta_1(\varepsilon)$$

On the circle L_ε we have

$$\sigma_{x_1}{}^\circ=\frac{d}{\varepsilon^2}\sum_{j=1}^{2}M_j\Pi_jA_j{}^\circ\frac{\cos^2\varphi-M_j{}^2\sin^2\varphi}{r_{j\varphi}^4}(t_0-t)$$

$$\sigma_{y_1}=-\frac{d}{\varepsilon^2}\sum_{j=1}^{2}M_j{}^{-1}\Pi_jA_j{}^\circ\frac{\cos^2\varphi-M_j{}^2\sin^2\varphi}{r_{j\varphi}^4}(t_0-t) \tag{5.7}$$

$$\tau_{xy_1}=\frac{d}{\varepsilon^2}\sum_{j=1}^{2}M_j\Pi_jA_j{}^\circ\frac{2\sin\varphi\cos\varphi}{r_{j\varphi}^4}(t_0-t)$$

From Hooke's law it follows that

$$\sigma_x(x_0,y_0,t)=a\frac{\partial u}{\partial x_0}+(c-d)\frac{\partial v}{\partial y_0},\qquad \sigma_y(x_0,y_0,t)=(c-d)\frac{\partial u}{\partial x_0}+a\frac{\partial v}{\partial y_0}$$

$$\tau_{xy}(x_0,y_0,t)=d\left(\frac{\partial u}{\partial y_0}+\frac{\partial v}{\partial x_0}\right) \tag{5.8}$$

We indicate next the values of $u_1{}^\circ$ and $v_1{}^\circ$ on L_ε

$$u_1{}^\circ=-\frac{c}{\varepsilon}\sum_{j=1}^{2}M_jA_j{}^\circ\frac{\cos\varphi}{r_{j\varphi}^2}(t_0-t),\qquad v_1{}^\circ=-\frac{1}{\varepsilon}\sum_{j=1}^{2}M_jB_j{}^\circ\frac{\sin\varphi}{r_{j\varphi}^2}(t_0-t) \tag{5.9}$$

By the use of Equations (5.5) and (5.7) we obtain

$$K_1=\varepsilon^2\int_0^{2\pi}(\sigma_{x_1}{}^\circ\cos^2\varphi+\tau_{xy_1}{}^\circ\sin\varphi\cos\varphi)\,d\varphi=$$

$$=d\sum_{j=1}^{2}M_j\Pi_jA_j{}^\circ\left[\int_0^{2\pi}\frac{\cos^4\varphi-M_j{}^2\sin^2\varphi\cos^2\varphi}{r_{j\varphi}^4}d\varphi+\int_0^{2\pi}\frac{2\sin^2\varphi\cos^2\varphi}{r_{j\varphi}^4}d\varphi\right](t_0-t)=$$

$$=2\pi d\sum_{j=1}^{2}\frac{\Pi_jA_j{}^\circ}{1+M_j}(t_0-t) \tag{5.10}$$

Analogous calculations give

$$K_2=\varepsilon^2\int_0^{2\pi}(\tau_{yx_1}{}^\circ\sin\varphi\cos\varphi+\sigma_{y_1}{}^\circ\sin^2\varphi)\,d\varphi=2\pi d\sum_{j=1}^{2}\frac{M_j{}^{-1}\Pi_jA_j{}^\circ}{1+M_j}(t_0-t) \tag{5.11}$$

We turn now to the third and fourth terms of Equation (5.6). We separate and compute the coefficients of $\partial u/\partial x_0$ and $\partial v/\partial y_0$, which are contained in these terms. On the basis of Equations (5.5), we obtain

$$K_1' = 2\pi\gamma d \sum_{j=1}^{2} \frac{M_j \Pi_j}{1+M_j}\left[\frac{a}{a-M_j^2 d} + \frac{c-d}{cM_j}\right](t_0 - t)$$
$$K_2' = 2\pi\gamma d \sum_{j=1}^{2} \frac{M_j \Pi_j}{1+M_j}\left[\frac{c-d}{a-M_j^2 d} + \frac{a}{cM_j}\right](t_0 - t) \tag{5.12}$$

Equation (5.6) can be rewritten in the form

$$\int_0^{t_0-\eta(\varepsilon)} \left[(K_1+K_1')\frac{\partial u}{\partial x_0} + (K_2+K_2')\frac{\partial v}{\partial y_0}\right] dt = D_{1\varepsilon} + \eta_1(\varepsilon) \tag{5.13}$$

We have

$$K_1 + K_1' = 2\pi \frac{\gamma d}{c} \sum_{j=1}^{2} \frac{\Pi_j}{(1+M_j)}\left\{\frac{d\Pi_j}{a-M_j^2 d} + \frac{acM_j}{a-M_j^2 d} + (c-d)\right\}(t_0 - t) =$$
$$= 2\pi\gamma\, da \sum_{j=1}^{2} \frac{\Pi_j}{a-M_j^2 d}(t_0 - t) = 2\pi a\,(t_0 - t) \tag{5.14}$$

Analogously

$$K_2 + K_2' = 2\pi \frac{\gamma d}{c} \sum_{j=1}^{2} \frac{\Pi_j}{1+M_j}\left[\frac{\Pi_j}{(a-M_j^2 d)M_j} + \frac{c(c-d)M_j}{d(a-M_j^2 d)} + \frac{a}{d}\right](t_0 - t) =$$
$$= 2\pi\gamma \frac{ad}{c} \sum_{j=1}^{2} \Pi_j (t_0 - t) = 2\pi(c+d)(t_0 - t) \tag{5.15}$$

Equation (5.13) takes on the form

$$2\pi \int_0^{t_0-\eta(\varepsilon)} \left[a\frac{\partial u}{\partial x_0} + (c+d)\frac{\partial v}{\partial y_0}\right](t_0 - t)\, dt = D_{1\varepsilon} + \eta_1(\varepsilon)$$

where $\eta_1(\varepsilon)$ and ε vanish simultaneously. Letting ε tend to zero, we obtain in the limit the first auxiliary equation corresponding to the first fundamental solution

$$2\pi \int_0^{t_0} \left[a\frac{\partial u}{\partial x_0} + (c+d)\frac{\partial v}{\partial y_0}\right](t_0 - t)\, dt = D_1 \tag{5.16}$$

Here

$$D_1 = -\iiint\limits_T (u_1^\circ X + v_1^\circ Y)\, dx\, dy\, dt - \iint\limits_{S_1} (\tau_{xy} u_1^\circ + \sigma_y v_1^\circ)\, dx dt +$$
$$+ \rho \iint\limits_{S_2} \left(\frac{\partial u}{\partial t} u_1^\circ + \frac{\partial u}{\partial t} v_1^\circ - \frac{\partial u_1^\circ}{\partial t} u - \frac{\partial v_1^\circ}{\partial t} v \right) dx\, dy \tag{5.17}$$

In an analogous manner, by applying the Green-Volterra formula to the required solution and the remaining fundamental solution, we obtain the second and third auxiliary relationships

$$\begin{aligned} 2\pi \int\limits_0^{t_0} \left[(c+d) \frac{\partial u}{\partial x_0} + a \frac{\partial v}{\partial y_0} \right] (t_0 - t)\, dt = D_2 \\ 2\pi \int\limits_0^{t_0} d \left(\frac{\partial u}{\partial y_0} - \frac{\partial v}{\partial x_0} \right) (t_0 - t)\, dt = D_3 \end{aligned} \tag{5.18}$$

where D_2 and D_3 are obtained from D_1 by a change of the functions u_1°, v_1° to u_2°, v_2° and u_3°, v_3°, respectively. To complete the problem we rewrite Equations (1.1) in the form

$$\begin{aligned} \frac{\partial}{\partial x_0} \left[a \frac{\partial u}{\partial x_0} + (c+d) \frac{\partial v}{\partial y_0} \right] + \frac{\partial}{\partial y_0} \left[d \left(\frac{\partial u}{\partial y_0} - \frac{\partial v}{\partial x_0} \right) \right] + X = \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial}{\partial y_0} \left[(c+d) \frac{\partial u}{\partial x_0} + a \frac{\partial v}{\partial y_0} \right] - \frac{\partial}{\partial x_0} \left[d \left(\frac{\partial u}{\partial y_0} - \frac{\partial v}{\partial x_0} \right) \right] + Y = \rho \frac{\partial^2 v}{\partial t^2} \end{aligned} \tag{5.19}$$

Differentiating Equations (5.17) and (5.18) with respect to the corresponding arguments, collecting terms, and integrating by parts, we obtain, with the aid of Equations (5.19)

$$\begin{aligned} u(x_0, y_0, t_0) &= u_0(x_0, y_0) + u_0'(x_0, y_0)\, t_0 + \frac{1}{\rho} \int\limits_0^{t_0} X(x_0, y_0, t)(t_0 - t)\, dt + \\ &\quad + \frac{1}{2\pi\rho} \left(\frac{\partial D_1}{\partial x_0} + \frac{\partial D_3}{\partial y_0} \right) \\ v(x_0, y_0, t_0) &= v_0(x_0, y_0) + v'_0(x_0, y_0)\, t_0 + \frac{1}{\rho} \int\limits_0^{t_0} Y(x_0, y_0, t)(t_0 - t)\, dt + \\ &\quad + \frac{1}{2\pi\rho} \left(\frac{\partial D_2}{\partial y_0} - \frac{\partial D_3}{\partial x_0} \right) \end{aligned} \tag{5.20}$$

These formulas give the solution of the problem that has been posed in closed form. Furthermore, they generalize the known result of Sobolev [2] relating to the isotropic body. Analogous results may also be obtained in a more general case of anisotropy - for example, in the case of four elastic constants [3].

6. Effect of point sources. We turn now to the investigation of the effect of various sorts of sources of oscillation, in particular, to the action of an instantaneous impulse on an unbounded anisotropic plane. We assume that up to the onset of the disturbance the medium is at rest

$$u_0(x, y) = v_0(x, y) = 0, \qquad u'_0(x, y) = v'_0(x, y) = 0 \tag{6.1}$$

Then, in accordance with (5.20), we obtain for an unbounded plane

$$\begin{aligned} u(x_0, y_0, t) &= \frac{1}{\rho}\int_0^{t_0} X(x_0, y_0, t)(t_0 - t)\,dt + \frac{1}{2\pi\rho}\left(\frac{\partial D_1}{\partial x_0} + \frac{\partial D_3}{\partial y_0}\right) \\ v(x_0, y_0, t) &= \frac{1}{\rho}\int_0^{t_0} Y(x_0, y_0, t)(t_0 - t)\,dt + \frac{1}{2\pi\rho}\left(\frac{\partial D_2}{\partial y_0} - \frac{\partial D_3}{\partial x_0}\right) \end{aligned} \tag{6.2}$$

where

$$D_k = -\iiint_T (Xu_k^\circ + Yv_k^\circ)\,dx\,dy\,dt, \qquad u_k^\circ = u_k^{\circ\circ}, \qquad v_k^\circ = v_k^{\circ\circ} \tag{6.3}$$

Here T is a portion of the xyt space, bounded by the large surface of the characteristic cone, constructed at the point (x_0, y_0, t_0), and the plane $t = 0$. Because of the singularities u_k°, v_k°, it is not possible in (6.2) to directly insert the differentiation sign under the integration sign in the calculation of the derivatives of D_k with respect to x_0 and y_0. However, in order to compute these derivatives, we may represent D_k in the form

$$D_k = -\iiint_{T-T'_\varepsilon} (Xu_k^\circ + Yv_k^\circ)\,dx\,dy\,dt - \iiint_{T'_\varepsilon} (Xu_k^\circ + Yv_k^\circ)\,dx\,dy\,dt \tag{6.4}$$

where T'_ε is a circular cylinder of radius ε and height $t_0 - \eta(\varepsilon)$. Hence, in the first integral, the interior boundary of the region of integration does not depend on x_0 and y_0. On the other hand, the external boundary of this region can also be made independent of the indicated arguments by virtue of the properties of the fundamental solutions. Hence, in the differentiation of the first term of (6.4) with respect to x_0 and y_0 the derivative may be inserted under the integral sign and (6.2) may be rewritten in the form

$$u(x_0, y_0, t_0) = \frac{1}{\rho}\int_0^{t_0} X(x_0, y_0, t)(t_0 - t)\,d\tau - \frac{1}{2\pi\rho}\iiint_{T-T'_\varepsilon}\left[\left(\frac{\partial u_1^\circ}{\partial x_0} + \frac{\partial u_3^\circ}{\partial y_0}\right)X + \right.$$

$$\left. + \left(\frac{\partial v_1^\circ}{\partial x_0} + \frac{\partial v_3^\circ}{\partial y_0}\right)Y\right]d\tau - \frac{1}{2\pi\rho}\int_0^{t_0-\eta(\varepsilon)}\left[\frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} Xu_1^\circ\,d\sigma + \frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} Xu_3^\circ\,d\sigma + \right.$$

$$+\frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} Y v_1^\circ\, d\sigma + \frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} Y v_3^\circ\, d\sigma\Big]dt \tag{6.5}$$

$$v(x_0, y_0, t_0) = \frac{1}{\rho}\int_0^{t_0} Y(x_0, y_0, t)(t_0 - t)\,dt - \frac{1}{2\pi\rho}\iiint_{T-T_\varepsilon}\Big[\Big(\frac{\partial u_2^\circ}{\partial y_0} - \frac{\partial u_3^\circ}{\partial x_0}\Big)X +$$

$$+\Big(\frac{\partial v_2^\circ}{\partial y_0} - \frac{\partial v_3^\circ}{\partial x_0}\Big)Y\Big]d\tau - \frac{1}{2\pi\rho}\int_0^{t_0-\eta(\varepsilon)}\Big[\frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} X u_2^\circ\, d\sigma - \frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} X u_3^\circ\, d\sigma +$$

$$+\frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} Y v_2^\circ\, d\sigma - \frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} Y v_3^\circ\, d\sigma\Big]dt$$

We calculate next the terms containing an integral over σ_ε. For small ε the functions u_k°, v_k° can be replaced by the estimates (4.6), (4.9), and (4.11). This allows one to use certain results from the theory of the logarithmic potential in the calculations. The functions

$$x'/r_j'^2, \qquad y_j'/r_j'^2, \qquad y_j' = y_j - y_{j0}, \qquad r_j'^2 = x'^2 + y_j'^2 \tag{6.6}$$

are harmonic on the plane xy_j, where $y_j = M_j y$. On this plane we have

$$\begin{aligned}
\frac{\partial}{\partial x_0}\iint_{\sigma_{j\varepsilon}} \rho^\circ(x, y_j)\ln\frac{1}{r'_j}\,dx\,dy_j &= \iint_{\sigma_{j\varepsilon}} \rho^\circ(x, y_j)\frac{x - x_0}{r_j'^2}\,dx\,dy_j \\
\frac{\partial}{\partial y_{j0}}\iint_{\sigma_{j\varepsilon}} \rho^\circ(x, y_j)\ln\frac{1}{r'_j}\,dx\,dy_j &= \iint_{\rho_{j\varepsilon}} \rho^\circ(x, y_j)\frac{y_j - y_{j0}}{r_j'^2}\,dx\,dy_j
\end{aligned} \tag{6.7}$$

$$\frac{\partial^2}{\partial x_2^\circ}\iint_{\sigma_{j\varepsilon}} \rho^\circ \ln\frac{1}{r'_j}\,dx\,dy_j + \frac{\partial^2}{\partial y^2_{j0}}\iint_{\sigma_{j\varepsilon}} \rho^\circ \ln\frac{1}{r'_j}\,dx\,dy_j = -2\pi\rho^\circ(x_0, y_{j0}) \tag{6.8}$$

where $\sigma_{j\varepsilon}$ is a region bounded by the ellipse $x^2 + y_j^2/M_j^2 = \varepsilon^2$. We consider the sum

$$(t_0 - t)N_1 = \frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} X u_1^\circ dx dy + \frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} X u_3^\circ dx\,dy \tag{6.9}$$

On the basis of (4.6) and (4.11) we have

$$N_1 = -c\sum_{j=1}^{2} M_j A_j^\circ\Big[\frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} X\frac{x - x_0}{r_j'^2}\,dx\,dy + \frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} X\frac{y - y_0}{r_j'^2}\,dx\,dy\Big] =$$

$$= -c\sum_{j=1}^{2} A_j^\circ\Big[\frac{\partial^2}{\partial x_0^2}\iint_{\sigma_{j\varepsilon}} X\Big(x, \frac{y_j}{M_j}, t\Big)\ln\frac{1}{r'_j}\,dx\,dy_j +$$

$$+\frac{\partial^2}{\partial y_{j0}^2}\iint_{\sigma_{j\varepsilon}} X\left(x,\frac{y_j}{M_j},t\right)\ln\frac{1}{r'_j}\,dx\,dy_j\Big] = 2\pi c X(x_0,y_0,t)\sum_{j=1}^{2}A_j^\circ = 2\pi X(x_0,y_0,t) \tag{6.10}$$

We next compute the sum

$$(t_0-t)\,N_1' = \frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} Y v_1^\circ\,dx\,dy + \frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} Y v_2^\circ\,dx\,dy \tag{6.11}$$

We have

$$N_1' = \sum_{j=1}^{2} B_j^\circ\left[M_j\frac{\partial}{\partial x_0}\iint_{\sigma_\varepsilon} Y\frac{y-y_0}{r'^2_j}\,dx\,dy - M_j^{-1}\frac{\partial}{\partial y_0}\iint_{\sigma_\varepsilon} Y\frac{x-x_0}{r'^2_j}\,dx\,dy\right]=$$

$$= \sum_{j=1}^{2} M_j^{-1}B_j^\circ\left[\frac{\partial^2}{\partial x_0\partial y_{j0}}\iint_{\sigma_{j\varepsilon}} Y\ln\frac{1}{r'_j}\,dx\,dy_j - \frac{\partial^2}{\partial y_{j0}\partial x_0}\iint_{\sigma_{j\varepsilon}} Y\ln\frac{1}{r'_j}\,dx\,dy_j\right] = 0 \tag{6.12}$$

Hence, by means of the results that have been obtained, we find after a passage to the limit that

$$u(x_0,y_0,t_0) = -\frac{1}{2\pi\rho}\iiint_T\left[\left(\frac{\partial u_1^\circ}{\partial x_0}+\frac{\partial u_2^\circ}{\partial y_0}\right)X + \left(\frac{\partial v_1^\circ}{\partial x_0}+\frac{\partial v_2^\circ}{\partial y_0}\right)Y\right]dx\,dy\,dt$$

Analogously we obtain

$$v(x_0,y_0,t_0) = -\frac{1}{2\pi\rho}\iiint_T\left[\left(\frac{\partial u_2^\circ}{\partial y_0^\circ}-\frac{\partial u_2^\circ}{\partial x_0}\right)X + \left(\frac{\partial v_2^\circ}{\partial y_0}-\frac{\partial v_2^\circ}{\partial x_0}\right)Y\right]dx\,dy\,dt \tag{6.13}$$

The problem of an applied concentrated impulse is now solved in the same way as in the case of an isotropic medium. We consider the sequence of functions X_n and Y_n. These are different from zero in a certain small region T_n whose dimensions go to zero with increasing n. We require further that for arbitrary n there should occur the equalities

$$\iiint_{T_n} X_n(x,y,t)\,dxdydt = P, \qquad \iiint_{T_n} Y_n(x,y,t)\,dx\,dy\,dt = Q \tag{6.14}$$

where P and Q are independent of n. Correspondingly, we have a second sequence

$$u_n(x_0,y_0,t_0) = -\frac{1}{2\pi\rho}\iiint_{T_n}\left[\left(\frac{\partial u_1^\circ}{\partial x_0}+\frac{\partial u_2^\circ}{\partial y}\right)X_n + \left(\frac{\partial v_1^\circ}{\partial x_0}+\frac{\partial v_2^\circ}{\partial y_0}\right)Y_n\right]dx\,dy\,dt$$

$$v_n(x_0, y_0, t_0) = -\frac{1}{2\pi\rho}\iiint_{T_n}\left(\frac{\partial u_2^\circ}{\partial y_0} - \frac{\partial u_3^\circ}{\partial x_0}\right)X_n + \left(\frac{\partial v_2^\circ}{\partial y_0} - \frac{\partial v_3^\circ}{\partial x_0}\right)Y_n\Big]\,dx\,dy\,dt \tag{6.15}$$

Making use of the theorem of the mean and letting n tend to infinity, we obtain, in the limit, the solution of the problem

$$\begin{aligned} u(x_0, y_0, t_0) &= -\frac{P}{2\pi\rho}\left(\frac{\partial u_1^\circ}{\partial x_0} + \frac{\partial u_3^\circ}{\partial y_0}\right) - \frac{Q}{2\pi\rho}\left(\frac{\partial v_1^\circ}{\partial x_0} + \frac{\partial v_3^\circ}{\partial y_0}\right) \\ v(x_0, y_0, t_0) &= -\frac{P}{2\pi\rho}\left(\frac{\partial u_2^\circ}{\partial y_0} - \frac{\partial u_3^\circ}{\partial x_0}\right) - \frac{Q}{2\pi\rho}\left(\frac{\partial v_2^\circ}{\partial y_0} - \frac{\partial v_3^\circ}{\partial x_0}\right) \end{aligned} \tag{6.16}$$

where all of the functions on the right are to be taken at $x = y = t = 0$. If, for example, $Q = 0$, then by using the values of u_k°, v_k° we find

$$\begin{aligned} u(x_0, y_0, t_0) &= \frac{P}{2\pi\rho}\sum_{j=1}^{2}\operatorname{Re}\frac{ic\theta_j\lambda_j}{\delta'_j}\left(\theta_j\omega_{1j}^{\circ\circ} - \lambda_j\omega_{3j}^{\circ\circ}\right) \\ v(x_0, y_0, t_0) &= -\frac{P}{2\pi\rho}\sum_{j=1}^{2}\operatorname{Re}\frac{ic\theta_j\lambda_j}{\delta'_j}\left(\lambda_j\omega_{2j}^{\circ\circ} + \theta_j\omega_{3j}^{\bullet\bullet}\right) \end{aligned} \tag{6.17}$$

where one should take into consideration the equations

$$\delta_j = t_0 - \theta_j x_0 + \lambda_j(\theta_j)\,y_0 = 0, \qquad \delta'_j = -x_0 + \lambda'_j(\theta_j)\,y_0$$

It is easy to verify directly that the functions (6.17) satisfy the equations of motion in the absence of body forces. Hence they give the solution of the problem of the action of an instantaneous impulse applied to an anisotropic plane along the direction of the x-axis. That is, Formulas (6.16) give the solution of the problem of the action of an instantaneous impulse, with components P and Q, applied at the origin of the reference axes on an anisotropic plane.

It is easy to show that the inequality $c < a - d$ is not essential. However, these constants should satisfy the inequality $c < a + d$. The latter is the condition of hyperbolicity of the system of Equations (1.1). Setting $c = a - d$, we arrive at the solution for the isotropic medium which was found by Sobolev.

BIBLIOGRAPHY

1. Sveklo, V.A., K resheniiu dinamicheskich zadach ploskoi teorii uprugosti dlia anizotropnogo tela (On the solution of dynamic problems in the plane theory of elasticity for an anisotropic medium). *PMM* Vol. 25, No. 5, 1961.

2. Frank, F. and Mises, R., *Differentsial'nye i integral'nye uravneniia matematicheskoi fiziki (Differential and Integral Equations of Mathematical Physics)*. ONTI, 1937.

3. Sveklo, V.A., Uprugie kolebaniia anizotropnogo tela (Elastic vibrations of an anisotropic body). *Uchen. zap. Leningr. un-ta*, No. 17, 1949.

Translated by E.E.Z.